ORC 69-25
AUGUST 1969

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This research has been partially supported by the Office of Naval Research under Contract Nonr-3656(18) and the National Science Foundation under Grant GK-1684 with the University of California.

[+]Paper presented at N.A.T.O. Conference on Reliability at Turino, Italy, June 30 - July 4, 1969.

ABSTRACT

Several nonparametric estimators of the failure rate function, assuming that the true unknown failure rate is increasing, are proposed and their asymptotic properties compared. Finite sample comparisons are made using Monte Carlo methods. Recommendations are made for the benefit of potential users.

1. INTRODUCTION

In many statistical studies involving failure data, biometric mortality data, and actuarial mortality data, the failure rate $r(x) = f(x)/[1 - F(x)]$, (corresponding to a lifetime distribution with density $f(x)$ and distribution function $F(x)$), is of prime importance. A problem of considerable interest, therefore, is the estimation of the failure rate function from a sample of n independent identically distributed lifetimes. Many observed failure rate function estimators seem to first decrease and then increase (are U-shaped) or simply increase or decrease. For simplicity, we will confine attention to *monotone* failure rate functions. (The U-shaped case can be estimated by suitably modifying the monotonic estimators, see for example Bray, Crawford and Proschan [4] and Barlow [1, pp. 60-61].)

In most cases, of course, the mortality data will be incomplete. Hence, we require estimators which can cope with incomplete data and which will, in some sense, be efficient. The efficiency of a method may be expressed by its mean square error, and one wants naturally an estimate of high efficiency. We will compare several nonparametric estimators of the failure rate function on the basis of their asymptotic mean square error for large sample sizes, as well as their Monte Carlo mean square error computed for small sample sizes.

Parametric and nonparametric methods for estimating the failure rate function are discussed in detail by Grenander [5]. He also characterizes the maximum likelihood estimate (MLE) of the failure rate function under the increasing failure rate (IFR) assumption. The MLE can be easily computed even for very incomplete data (withdrawals may be allowed for example). If a total of n items are exposed to risk, failures are observed at times

$$Z_1 \le Z_2 \le \dots \le Z_k \qquad (k \le n)$$

and $n(u)$ is the number of items exposed to risk at time u, then the MLE estimate for the failure rate, $r(t)$, can be expressed as a step function, where

$$(1.1)\qquad \hat{r}_n(t) = \begin{cases} 0 & 0 \leq t < Z_1 \\ \hat{r}_n(Z_i) & Z_i \leq t < Z_{i+1} \\ \infty & t \geq Z_k \end{cases}$$

and

$$(1.2)\qquad \hat{r}_n(Z_i) = \underset{s \leq i}{\mathrm{Max}}\ \underset{t \geq i+1}{\mathrm{Min}} \left\{ \frac{t - s}{\sum_{j=s}^{t-1} \int_{Z_j}^{Z_{j+1}} n(u)du} \right\} .$$

Marshall and Proschan [6] proved that $\hat{r}_n(t)$ is strongly consistent in the complete sample case.

Prakasa Rao [7] has characterized the limiting distribution of $\hat{r}_n(t)$ assuming $r(t)$ increasing and $r'(t) > 0$. He shows that

$$(1.3)\qquad \mathcal{L}\left[n^{\frac{1}{3}} \left\{ \frac{2f(t)}{r'(t)r^2(t)} \right\}^{\frac{1}{3}} \left\{ \hat{r}_n(t) - r(t) \right\} \right] \to H(x)$$

where $H(x)$ is a distribution whose density is determined implicitly as a solution to the heat equation. This result enables us to make asymptotic comparisons with other nonparametric estimators of the failure rate function.

The main deficiencies of the maximum likelihood estimate as defined in (1.1) and (1.2) are:

(1) According to (1.3) the MLE converges like $n^{-\frac{1}{3}}$ while many estimators for parametric models converge much faster; e.g., like $n^{-\frac{1}{2}}$.

(2) Some failure times must be observed to use (1.2) whereas in practice we often know only the number of failures within prescribed intervals. However, this criticism is partially unfair since the MLE estimate for discrete IFR distributions could be used in this situation (see Marshall and Proschan [6], pp. 76-77).

Watson and Leadbetter [9] study nonparametric estimators for the failure rate function which perform better, asymptotically, than the IFR maximum likelihood estimator. However, their estimator will in general be very "ragged" and certainly not monotonic even though we may feel that the true failure rate function is monotonic. Let $0 \equiv w_{0,n} < w_{1,n} < w_{2,n} < \ldots$ be a subdivision of $[0,\infty)$. Let F_n be the empirical distribution function corresponding to an ordered sample

$$0 \equiv X_o < X_1 < X_2 < \ldots < X_n$$

from F ; i.e.,

$$F_n(x) = \begin{cases} 0 & x < X_1 \\ \frac{i}{n} & X_i \leq x < X_{i+1} \\ 1 & x \geq X_n . \end{cases}$$

Watson and Leadbetter [9] study the estimator

$$r_n(x) = \frac{F_n(w_{i+1,n}) - F_n(w_{i,n})}{(w_{i+1,n} - w_{i,n})[1 - F_n(w_{i,n})]}$$

where $w_{i,n} \leq x < w_{i+1,n}$. Let $w_{i+1,n} - w_{i,n} = cn^{-\alpha}$ for $\frac{1}{5} < \alpha < 1$. They show that asymptotically, as $n \to \infty$

$$\mathscr{L}\left\{\left[\frac{cf(x)}{r(x)}\right]^{\frac{1}{2}} n^{\frac{1-\alpha}{2}} [r_n(x) - r(x)]\right\} \to N(0,1) \tag{1.4}$$

where $N(0,1)$ stands for the normal distribution with mean 0 and variance 1.

If we compare (1.3) and (1.4), we see that $r_n(x)$ will be asymptotically more efficient than the MLE for IFR failure rate functions when $\alpha < \frac{1}{3}$. Intuitively, it seems reasonable that if we can "smooth" $r_n(x)$ so that it is monotonic, we will be able to improve on the IFR maximum likelihood estimate.

2. MONOTONIC REGRESSION ESTIMATORS

Let $0 \equiv w_{0,n} < w_{1,n} < \dots < w_{i,n} < \dots$ be a subdivision of $[0,\infty)$ and $\mu_n\{w_{j,n}\}$ a sequence of weights on $\left\{w_{j,n}\right\}_{j=0}^{\infty}$. Assume $w_{i,n} \leq x < w_{i+1,n}$. We consider two basic or initial estimators for the failure rate function, namely,

$$r_n(x) = \frac{F_n(w_{i+1,n}) - F_n(w_{i,n})}{(w_{i+1,n} - w_{i,n})[1 - F_n(w_{i,n})]} \tag{2.1}$$

and

$$\phi_n(x) = \frac{-\log\,[1 - F_n(w_{i,n})] + \log\,[1 - F_n(w_{i,n})]}{w_{i+1,n} - w_{i,n}}\ . \tag{2.2}$$

We call $\phi_n(x)$ the "graphical" estimator since it is motivated by the fact that

$$\int_{w_{i,n}}^{w_{i+1,n}} r(u)du = -\log\,[1 - F(w_{i+1,n})] + \log\,[1 - F(w_{i,n})]\ .$$

The monotone increasing regression estimators corresponding to (2.1) and (2.2) and weights $\mu\{w_{i,n}\} \geq 0$ for $i = 1,2, \dots$ are

$$\hat{r}_n(x) = \underset{t\geq i+1}{\text{Min}}\ \underset{s\leq i}{\text{Max}}\ \frac{\sum_{j=s}^{t-1} r_n(w_{j,n})\mu_n\{w_{j,n}\}}{\sum_{j=s}^{t-1} \mu_n\{w_{j,n}\}} \tag{2.3}$$

and

$$\hat{\phi}_n(x) = \underset{t\geq i+1}{\text{Min}}\ \underset{s\leq i}{\text{Max}}\ \frac{\sum_{j=s}^{t-1} \phi_n(w_{j,n})\mu_n\{w_{j,n}\}}{\sum_{j=s}^{t-1} \mu_n\{w_{j,n}\}}\ . \tag{2.4}$$

(Monotone decreasing regression estimators correspond to interchanging the Min and Max operations.) It is easy to verify that both $\hat{r}_n(x)$ and $\hat{\phi}_n(x)$ are nondecreasing functions of x. It can also be shown that if $r(x)$ is the true nondecreasing failure rate function, then, in a certain sense, $\hat{r}_n$ is closer than r_n to r and $\hat{\phi}_n$ is closer than ϕ_n to r.

We shall see that the maximum likelihood estimate for $r(x)$ suggests the *total time on test* weights. Suppose we take as our grid, the order statistics $0 \equiv X_0 < X_1 < \ldots < X_n$ and

$$\mu_n\{X_i\} = \frac{1}{n}(n - i)(X_{i+1} - X_i) = [1 - F_n(X_i)](X_{i+1} - X_i) .$$

Then $\hat{r}_n(x)$ becomes, for $X_i \leq x < X_{i+1}$,

$$\hat{r}_n(x) = \underset{t \geq i+1}{\text{Min}} \ \underset{s \leq i}{\text{Max}} \frac{\sum_{j=s}^{t-1} r_n(X_j)[1 - F_n(X_j)](X_{j+1} - X_j)}{\sum_{j=s}^{t-1} [1 - F_n(X_j)](X_{j+1} - X_j)} \tag{2.5}$$

or

$$\hat{r}_n(x) = \underset{t \geq i+1}{\text{Min}} \ \underset{s \leq i}{\text{Max}} \frac{t - s}{\sum_{j=s}^{t-1} (n - j)(X_{j+1} - X_j)} . \tag{2.6}$$

We recognize (2.6) as the IFR MLE (1.2) when we have a complete sample. Thus, the IFR MLE is a special case of a monotonic regression estimator. Note that $(n - i)(X_{i+1} - X_i)$ is the total time on test between the ith and $(i + 1)$st failure. For arbitrary girds (2.5) becomes

$$(2.7)\qquad \hat{r}_n(x) = \underset{t \geq i+1}{\text{Min}}\ \underset{s \leq i}{\text{Max}} \frac{F_n(w_{t,n}) - F_n(w_{s,n})}{\sum_{j=s}^{t-1} [1 - F_n(w_{j,n})](w_{j+1,n} - w_{j,n})} .$$

If we choose $\mu_n\{w_{i,n}\} = (w_{i+1,n} - w_{i,n})$ and use the graphical estimator, then (2.4) becomes

$$(2.8)\qquad \hat{\phi}_n(x) = \underset{t \geq i+1}{\text{Min}}\ \underset{s \leq i}{\text{Max}} \frac{-\log [1 - F_n(w_{t,n})] + \log [1 - F_n(w_{s,n})]}{w_{t,n} - w_{s,n}}$$

Although we consider only monotonic regression estimators of the form (2.7) or (2.8) it is clear that we have a whole zoo of possible estimators. We are at liberty to choose the basic estimators, $r_n(x)$, the grid $\left\{w_{i,n}\right\}_{i=1}^{\infty}$ and the weights $\mu_n\{w_{i,n}\} \geq 0$. In [2] it is shown that $\hat{r}_n(x)$ and $\hat{\phi}_n(x)$ converge to $r(x)$ as $n \to \infty$ under very general conditions on the grid size, etc.

3. ASYMPTOTIC RESULTS

In order to make asymptotic comparisons, we consider grids with spacings of the type $w_{i+1,n} - w_{i,n} = cn^{-\alpha}$ for $c > 0$ and $0 < \alpha < 1$.

It can be shown that the estimator, $r_n(x)$, given by (2.1) is asymptotically normal with mean

$$r(x) + \frac{1}{24}\frac{r(x)f''(x)}{f(x)} c^2 n^{-2\alpha} + 0(n^{-3\alpha}) \tag{3.1}$$

and variance

$$\frac{r^2(x)n^{\alpha-1}}{f(x)} + 0(n^{-1}) . \tag{3.2}$$

Hence, the mean square error (MSE) of $r_n(x)$ is approximately

$$\text{MSE } [r_n(x)] \doteq \frac{r^2(x)n^{\alpha-1}}{cf(x)} + \left[\frac{c^2 r(x)f''(x)}{24f(x)}\right]^2 n^{-4\alpha} \tag{3.3}$$

For $1/5 < \alpha < 1$

$$\frac{\sqrt{cf(x)}}{r(x)} n^{\frac{1-\alpha}{2}} [r_n(x) - r(x)] \tag{3.4}$$

has asymptotically a $N(0,1)$ distribution, while for $1/7 < \alpha < 1/5$

$$\frac{\sqrt{cf(x)}}{r(x)} n^{\frac{1-\alpha}{2}} \left[r_n(x) - r(x) - \frac{c^2 n^{-2\alpha} r(x)f''(x)}{24}\right] \tag{3.5}$$

is asymptotically $N(0,1)$.

It is show in [3] that for $0 < \alpha < 1/3$ *and* $r'(x) > 0$

$$\text{MSE } [\hat{r}_n(x)] \doteq \text{MSE } [r_n(x)]$$

asymptotically so that asymptotically the monotonic regression estimator is no more efficient than the basic estimator, $r_n(x)$. However, as we pointed out earlier, if the true failure rate is nondecreasing, the monotonic regression estimator will be "closer" to the true failure rate, at least for small sample sizes, than the basic estimator. However, if the true distribution is *exponential*, the monotonic regression estimator will converge like $n^{-\frac{1}{2}}$ for any choice of $\alpha (0 < \alpha \leq 1)$ [see [3]]. Note that in this case even the IFR MLE is superior to the basic estimator.

It is shown in [3], that the estimator $\phi_n(x)$ is asymptotically normal with mean

$$(3.6) \qquad r(x) + \frac{1}{24} r''(x) c^2 n^{-2\alpha} + 0(n^{-3\alpha})$$

and variance

$$(3.7) \qquad \frac{r^2(x) n^{\alpha-1}}{ncn^{-\alpha} f(x)} + 0(n^{-1}) .$$

For $1/5 < \alpha < 1$, $\phi_n(x)$ is asymptotically equivalent to $r_n(x)$. However, for $1/7 < \alpha \leq 1/5$

$$(3.8) \qquad \frac{\sqrt{cf(x)}}{r(x)} n^{\frac{1-\alpha}{2}} \left[\phi_n(x) - r(x) - \frac{c^2 r''(x)}{24} n^{-2\alpha}\right]$$

is asymptotically $N(0,1)$. Note that (3.5) and (3.8) differ in the bias term.

Asymptotically, for $0 < \alpha < 1/3$, $\hat{\phi}_n(x)$ and $\phi_n(x)$ are equivalent. However, the same remarks concerning $\hat{r}_n(x)$ apply to $\hat{\phi}_n(x)$. Therefore, if the true failure rate is nondecreasing, we would prefer $\hat{\phi}_n(x)$ to $\phi_n(x)$ with $0 < \alpha < 1/3$.

The mean square error of both $r_n(x)$ and $\phi_n(x)$ are minimized when we choose $\alpha = 1/5$. Therefore, on the basis of asymptotic considerations, we would recommend using either $\hat{r}_n(x)$ or $\hat{\phi}_n(x)$ with $\alpha = 1/5$. However, this is not

too helpful since the optimum choice of c would still depend on the true failure rate function which is of course unknown. The MLE takes care of this problem by choosing $w_{i,n} = X_i$, the ith order statistic from our sample. One might modify the MLE by choosing $w_{i,n} = X_{[in^\beta]}$ where [ ] denotes the greatest integer in the quantity within the brackets and $\beta = 1 - \alpha$. Hence,

$$X_{[(i+1)n^\beta]} - X_{[in^\beta]} = 0_p\left(\frac{n^\beta}{n}\right) = 0_p(n^{-\alpha}) ,$$

and by choosing $\alpha = 1/5$, we realize the recommended requirement.

4. NUMERICAL RESULTS

Asymptotic investigations indicate that we should use either $\hat{\phi}_n(x)$ or $\hat{r}_n(x)$ with $\alpha = 1/5$. However, for this choice of α , $\hat{\phi}_n(x)$ and $\hat{r}_n(x)$ are *not* asymptotically equivalent. They differ in their bias terms. In general, we will prefer $\hat{\phi}_n(x)$ if r is "nearly" linear in a neighborhood of x and $\hat{r}_n(x)$ if f is "nearly" linear in a neighborhood of x .

In order to choose between $\hat{\phi}_n(x)$ and $\hat{r}_n(x)$, several numerical investigations have been made. The most extensive numerical investigations have been conducted by Watson and Leadbetter [8] and [9]. They did not consider monotonic regression estimators except for the IFR MLE estimator. They obtained the best results from a "heuristic graphical" estimator. To obtain this estimator, they plot $-\log [1 - F_n(x)]$ against x . A smooth curve is then drawn through these points and its slope determined, perhaps just with a straight edge. This estimator, by its construction, does *not* come with formulae for its mean and variance. The estimator $\hat{\phi}_n(x)$ is similiar to this estimator except that the slopes of the tangent lines are now required to be increasing. Because of this similarity, we might expect the good results obtained for their heuristic estimator to carry over for $\hat{\phi}_n(x)$.

Tables I and II present the mean square error of the IFR MLE, $\hat{r}_n(x)$, and its mean value when the underlying failure distribution is exponential with unit mean. Since the estimator is always infinite beyond the largest observation, we have also included the number of infinities obtained in the simulations.

Tables III and IV present the mean square error and mean value of the MLE estimator assuming a U-shaped failure rate function when the underlying failure distribution is exponential with unit mean. In the case of an infinite estimator within the range of x values recorded, the estimator was assumed constant (not infinite) beyond the last order statistic for which the estimator was finite. This probably accounts for the large mean square error values for large x values.

Tables V through XIV compare the graphical estimator and the maximum likelihood estimator using the grid determined by order statistics when the true distribution is a Weibull distribution. On the basis of mean square error calculations, we can detect little difference in the two estimators.

These computer computations were performed by Tom Bray of the Boeing Scientific Research Laboratories.

TABLE I

MLE FOR INCREASING FAILURE RATE WHEN TRUE DISTRIBUTION IS THE UNIT EXPONENTIAL DISTRIBUTION

Number of Simulations = 3000

Sample Size = n = 25

X	MEAN VALUE OF $\hat{r}_n(x)$	MEAN SQUARE ERROR OF $\hat{r}_n(x)$	NUMBER OF INFINITIES
0.10	0.606914	0.210603	0
0.20	0.814584	0.117653	0
0.30	0.886149	0.902205E-01	0
0.40	0.927219	0.835712E-01	0
0.50	0.962705	0.841033E-01	0
0.60	0.993960	0.869380E-01	0
0.70	1.02032	0.918217E-01	0
0.80	1.04681	0.101941	0
0.90	1.07076	0.112549	0
1.00	1.09374	0.126824	0
1.10	1.12219	0.157527	0
1.20	1.15558	0.239723	0
1.30	1.19133	0.428387	1
1.40	1.25097	3.72891	2
1.50	1.24791	0.392483	6

TABLE II

MLE FOR INCREASING FAILURE RATE WHEN TRUE DISTRIBUTION IS THE UNIT EXPONENTIAL DISTRIBUTION

Number of Simulations = 250

Sample Size = n = 100

X	MEAN VALUE OF $\hat{r}_n(x)$	MEAN SQUARE ERROR OF $\hat{r}_n(x)$	NUMBER OF INFINITIES
0.20	0.919915	0.268983E-01	0
0.40	0.974432	0.176583E-01	0
0.60	1.01091	0.151868E-01	0
0.80	1.03226	0.199563E-01	0
1.00	1.05027	0.241104E-01	0
1.20	1.06319	0.265414E-01	0
1.40	1.07543	0.307885E-01	0
1.60	1.09109	0.364013E-01	0
1.80	1.12102	0.581159E-01	0
2.00	1.14981	0.797719E-01	0
2.20	1.18126	0.106373	0
2.40	1.22572	0.161607	0
2.60	1.27003	0.236557	0
2.80	1.30381	0.259973	2
3.00	1.43084	0.686589	2

TABLE III

MLE FOR U-SHAPED FAILURE RATE WHEN TRUE DISTRIBUTION IS THE UNIT EXPONENTIAL DISTRIBUTION

Number of Simulations = 150

Sample Size = n = 25

X	MEAN VALUE OF $\hat{r}_n(x)$	MEAN SQUARE ERROR OF $\hat{r}_n(x)$	NUMBER OF INFINITIES
0.05	1.29297	0.549797	0
0.10	1.08526	0.381039	0
0.15	0.959894	0.314845	0
0.20	0.937388	0.302800	0
0.25	0.890439	0.273362	0
0.30	0.877699	0.253101	0
0.35	0.864808	0.232931	0
0.40	0.835557	0.228444	0
0.45	0.828066	0.220882	0
0.50	0.788042	0.249237	0
0.60	0.758739	0.266121	0
0.70	0.782102	0.257277	0
0.80	0.836567	0.215058	0
0.90	0.820652	0.253918	0
1.00	0.824013	0.265950	0
1.20	0.922699	0.324050	0
1.40	1.04247	0.439205	0
1.60	1.17449	0.861991	0
1.80	1.63692	17.6990	2
2.00	1.83803	18.7647	3
2.20	4.56656	502.373	10
2.40	5.06498	514.575	17
2.60	5.74322	549.582	23
2.80	8.96341	1544.13	31
3.00	9.17613	1545.75	50
3.20	9.51944	1550.91	55
3.40	9.77481	1556.98	66
3.60	9.80845	1557.00	76
3.80	9.80993	1557.00	87
4.00	9.81614	1557.01	96

TABLE IV

MLE FOR U-SHAPED FAILURE RATE WHEN TRUE DISTRIBUTION IS THE UNIT EXPONENTIAL DISTRIBUTION

Number of Simulations = 100

Sample Size = n = 50

x	MEAN VALUE OF $\hat{r}_n(x)$	MEAN SQUARE ERROR OF $\hat{r}_n(x)$	NUMBER OF INFINITIES
0.05	1.16594	0.265775	0
0.10	1.04175	0.164489	0
0.15	1.00883	0.148645	0
0.20	0.994515	0.108994	0
0.25	0.961543	0.128702	0
0.30	0.956456	0.124807	0
0.35	0.888189	0.164545	0
0.40	0.860720	0.174514	0
0.45	0.852992	0.187309	0
0.50	0.845861	0.205879	0
0.60	0.849351	0.219402	0
0.70	0.940997	0.122661	0
0.80	0.943531	0.127397	0
0.90	0.915578	0.159765	0
1.00	0.910690	0.162759	0
1.20	0.936932	0.143381	0
1.40	0.952359	0.165201	0
1.60	1.00287	0.168174	0
1.80	1.06294	0.212498	0
2.00	1.10732	0.328394	0
2.20	1.13930	0.389655	0
2.40	1.46696	5.32901	2
2.60	1.50744	5.38283	2
2.80	1.70013	6.04579	2
3.00	1.97122	8.75948	6
3.20	2.65745	23.2704	14
3.40	2.77963	23.8924	21
3.60	2.86158	24.2406	27
3.80	2.91245	24.3000	29
4.00	2.98504	24.4582	34

TABLE V

$\alpha = 1$

MLE FOR INCREASING FAILURE RATE WHEN TRUE DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^{\alpha}}$$

Number of Simulations = 500

Sample Size = n = 20

x	MEAN BIAS	MEAN SQUARE ERROR OF $r_n(x)$	NUMBER OF INFINITIES
0.0	+ 0.00000	0.00000	0
0.2	- 0.17781	0.11695	0
0.4	+ 0.08053	0.08095	0
0.6	+ 0.00269	0.10185	0
0.8	+ 0.06714	0.13693	0
1.0	+ 0.12675	0.17096	0
1.2	+ 0.18478	0.26036	0
1.4	+ 0.23401	0.32004	1
1.6	+ 0.37319	0.90744	1
1.8	+ 0.48808	1.54212	7
2.0	+ 0.62947	4.67073	23

TABLE VI

$\alpha = 1$

GRAPHICAL ESTIMATOR OF FAILURE RATE FUNCTION WHEN TRUE DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^{\alpha}}$$

Number of Simulations = 500

Sample Size = n = 20

x	MEAN BIAS	MEAN SQUARED ERROR OF $r_n(x)$	NUMBER OF INFINITIES
0.0	+ 0.00000	0.00000	0
0.2	- 0.22220	0.11886	0
0.4	- 0.14343	0.08153	0
0.6	- 0.06969	0.09808	0
0.8	- 0.01284	0.12645	0
1.0	+ 0.04850	0.16330	0
1.2	+ 0.11063	0.25271	0
1.4	+ 0.17110	0.32801	1
1.6	+ 0.32720	0.96436	1
1.8	+ 0.45788	1.63002	7
2.0	+ 0.60703	4.76836	23

TABLE VII

$\alpha = 1.5$

MLE FOR INCREASING FAILURE RATE WHEN TRUE DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^{\alpha}}$$

Number of Simulations

Sample Size = n = 20

x	MEAN BIAS	MEAN SQUARE ERROR OF $r_n(x)$	NUMBER OF INFINITIES
0.0	+ 0.00000	0.00000	0
0.2	+ 0.04326	0.05232	0
0.4	+ 0.00724	0.09421	0
0.6	+ 0.04241	0.14567	0
0.8	+ 0.11443	0.25389	0
1.0	+ 0.21644	0.43585	0
1.2	+ 0.35952	1.06299	0
1.4	+ 0.68761	3.39264	2
1.6	+ 1.15985	23.00837	24
1.8	+ 1.02670	9.71217	86
2.0	+ 1.24338	103.99067	159

TABLE VIII

$\alpha = 1.5$

GRAPHICAL ESTIMATOR OF FAILURE RATE FUNCTION WHEN TRUE DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^{\alpha}}$$

Number of Simulations = 500

Sample Size = n = 20

x	MEAN BIAS	MEAN SQUARE ERROR OF $r_n(x)$	NUMBER OF INFINITIES
0.0	+ 0.00000	0.00000	0
0.2	+ 0.04180	0.04764	0
0.4	- 0.01285	0.08097	0
0.6	+ 0.00005	0.12260	0
0.8	+ 0.05764	0.22584	0
1.0	+ 0.15529	0.41857	0
1.2	+ 0.31463	1.08971	0
1.4	+ 0.68591	3.67826	2
1.6	+ 1.18606	23.28767	24
1.8	+ 1.04786	9.86327	86
2.0	+ 1.26325	104.11156	159

TABLE IX

$\alpha = 2.0$

MLE FOR INCREASING FAILURE RATE WHEN TRUE DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^{\alpha}}$$

Number of Simulations = 500

Sample Size = n = 20

x	MEAN BIAS	MEAN SQUARE ERROR OF $r_n(x)$	NUMBER OF INFINITIES
0.0	+ 0.00000	0.00000	0
0.2	+ 0.08906	0.03585	0
0.4	+ 0.07154	0.09127	0
0.6	+ 0.07192	0.18479	0
0.8	+ 0.15319	0.36775	0
1.0	+ 0.32118	0.84297	0
1.2	+ 0.57333	2.19050	1
1.4	+ 1.45527	26.21926	17
1.6	+ 1.16589	9.95119	113
1.8	+ 1.52419	39.64151	223
2.0	+ 0.12816	2.52929	342

TABLE X

$\alpha = 2.0$

GRAPHICAL ESTIMATOR OF FAILURE RATE FUNCTION WHEN TRUE DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^{\alpha}}$$

Number of Simulations = 500

Sample Size = n = 20

x	MEAN BIAS	MEAN SQUARE ERROR OF $r_n(x)$	NUMBER OF INFINITIES
0.0	+ 0.00000	0.00000	0
0.2	+ 0.09201	0.03649	0
0.4	+ 0.06967	0.08389	0
0.6	+ 0.04531	0.15483	0
0.8	+ 0.10356	0.31729	0
1.0	+ 0.26524	0.81702	0
1.2	+ 0.54895	2.30813	1
1.4	+ 1.57024	28.33272	17
1.6	+ 1.22317	10.29591	113
1.8	+ 1.58241	39.89294	223
2.0	+ 0.13982	2.62531	342

TABLE XI

$\alpha = 2.5$

MLE FOR INCREASING FAILURE RATE WHEN TRUE DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^{\alpha}}$$

Number of Simulations = 500

Sample Size = n = 20

x	MEAN BIAS	MEAN SQUARE ERROR OF $r_n(x)$	NUMBER OF INFINITIES
0.0	+ 0.00000	0.00000	0
0.2	+ 0.06873	0.02436	0
0.4	+ 0.11064	0.07822	0
0.6	+ 0.09774	0.18706	0
0.8	+ 0.19814	0.49999	0
1.0	+ 0.43034	1.39276	0
1.2	+ 1.08370	8.84637	1
1.4	+ 2.47167	58.14368	68
1.6	+ 2.12447	78.06147	222
1.8	- 0.17729	7.53517	385
2.0	- 1.62648	4.42413	467

TABLE XII

$\alpha = 2.5$

GRAPHICAL ESTIMATOR OF FAILURE RATE FUNCTION WHEN TRUE DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^{\alpha}}$$

Number of Simulations = 500

Sample Size = n = 20

x	MEAN BIAS	MEAN SQUARE ERROR OF $r_n(x)$	NUMBER OF INFINITIES
0.0	+ 0.00000	0.00000	0
0.2	+ 0.07079	0.02546	0
0.4	+ 0.11355	0.07525	0
0.6	+ 0.08569	0.16751	0
0.8	+ 0.15261	0.42956	0
1.0	+ 0.37411	1.35641	0
1.2	+ 1.10588	9.49925	1
1.4	+ 2.62338	69.11603	68
1.6	+ 2.22374	78.52692	222
1.8	- 0.16100	7.47486	385
2.0	- 1.60758	4.27771	467

TABLE XIII

$\alpha = 3.0$

MLE FOR INCREASING FAILURE RATE WHEN TRUE DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^{\alpha}}$$

Number of Simulations = 500

Sample Size = n = 20

x	MEAN BIAS	MEAN SQUARE ERROR OF $r_n(x)$	NUMBER OF INFINITIES
0.0	+ 0.00000	0.00000	0
0.2	+ 0.03376	0.01045	0
0.4	+ 0.11782	0.05387	0
0.6	+ 0.13871	0.21125	0
0.8	+ 0.20063	0.55209	0
1.0	+ 0.53841	2.08249	0
1.2	+ 1.85227	21.50619	5
1.4	+ 2.00208	25.27304	140
1.6	+ 0.89666	62.18391	354
1.8	+ 0.20132	100.25058	471
2.0	- 2.28445	6.30387	498

TABLE XIV

$\alpha = 3.0$

GRAPHICAL ESTIMATOR OF FAILURE RATE FUNCTION WHEN TRUE DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^{\alpha}}$$

Number of Simulations = 500

Sample Size = n = 20

x	MEAN BIAS	MEAN SQUARE ERROR OF $r_n(x)$	NUMBER OF INFINITIES
0.0	+ 0.00000	0.00000	0
0.2	+ 0.03467	0.01093	0
0.4	+ 0.12249	0.05486	0
0.6	+ 0.13366	0.19345	0
0.8	+ 0.16158	0.48076	0
1.0	+ 0.48233	2.03612	0
1.2	+ 1.95784	23.51841	5
1.4	+ 2.17050	27.00391	140
1.6	+ 0.93123	62.01564	354
1.8	+ 0.19225	100.29024	471
2.0	- 2.28445	6.30387	498

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)		
1. ORIGINATING ACTIVITY (Corporate author) University of California, Berkeley	2a. REPORT SECURITY CLASSIFICATION Unclassified	2b. GROUP
3. REPORT TITLE COMPARISON OF SEVERAL NONPARAMETRIC ESTIMATORS OF THE FAILURE RATE FUNCTION		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Research Report		
5. AUTHOR(S) (First name, middle initial, last name) Richard E. Barlow and Willem R. van Zwet		
6. REPORT DATE August 1969	7a. TOTAL NO. OF PAGES 27	7b. NO. OF REFS 9
8a. CONTRACT OR GRANT NO. GK-1684 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) ORC 69-25 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
10. DISTRIBUTION STATEMENT This document has been approved for public release and sale; its distribution is unlimited.		
11. SUPPLEMENTARY NOTES Also supported by the Office of Naval Research under Contract Nonr-3656(18).	12. SPONSORING MILITARY ACTIVITY NSF-GK-1684-Shephard, The National Science Foundation Washington, D.C. 20550	
13. ABSTRACT SEE ABSTRACT.		

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

Unclassified

Security Classification

Unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Estimation						
Monotone Failure Rate						
Monotonic Regression						
Total Time on Test Weights						

DD FORM 1 NOV 65 1473 (BACK)

S/N 0101-807-6821

Unclassified

Security Classification

A-31409